ELSEVIER
International Journal of Solids and Structures 41 (2004) 279–294
INTERNATIONAL JOURNAL OF SOLIDS and STRUCTURES
www.elsevier.com/locate/ijsolstr

Exact solutions for buckling and vibration of stepped rectangular Mindlin plates

Y. Xiang [a,b,*], G.W. Wei [c]

[a] School of Engineering and Industrial Design, University of Western Sydney, Kingswood Campus, Locked Bag 1797, Penrith, South DC, NSW 1797, Australia
[b] Centre for Construction Technology and Research, University of Western Sydney, Penrith, South DC, NSW 1797, Australia
[c] Department of Mathematics, Michigan State University, East Lansing, MI 48824, USA

Received 20 June 2001; received in revised form 24 May 2002

Abstract

This paper presents the first-known exact solutions for buckling and vibration of stepped rectangular Mindlin plates with two opposite edges simply supported and the remaining two edges being either free, simply supported or clamped. The general Levy type solution method and a domain decomposition technique are employed to develop an analytical approach to deal with the stepped rectangular Mindlin plates. Exact buckling loads and vibration frequencies are obtained for two-, three- and four-stepped Mindlin plates with varying step thickness ratios. The influence of the step length ratios, step thickness ratios and the number of steps on the buckling and vibration behaviour of square and rectangular Mindlin plates is examined. The presented exact results may serve as benchmark solutions for such plates.

1. Introduction

Plates with varying thickness are extensively used in modern structures due to their unique functions. For example, stepped plates possess a number of attractive features, such as material saving, weight reduction, stiffness enhancing, designated strengthening, fundamental vibration frequency increasing, etc. With the availability of inexpensive and high performance computers, theoretical analysis is frequently employed to optimize stepped plates in practical engineering designs. In particular, buckling and free vibration analysis of stepped plates has attracted much attention in the past few decades. A variety of theoretical approaches have been formulated for this class of problems. These approaches may be applied to study varying thickness plates where the plate thickness is allowed to vary either as piecewise constant step functions (e.g. Chopra, 1974; Yuan and Dickinson, 1992; Lam and Amrutharaj, 1995; Guo et al., 1997; Eisenberger and Alexandrov, 2000; Ju et al., 1995; Cheung et al., 2000), a linear function (e.g. Wittrick and

* Corresponding author. Address: School of Engineering and Industrial Design, University of Western Sydney, Kingswood Campus, Locked Bag 1797, Penrith, South DC, NSW 1797, Australia. Tel.: +61-2-47360395; fax: +61-2-47360833.
E-mail address: y.xiang@uws.edu.au (Y. Xiang).

0020-7683/$ - see front matter
doi:10.1016/j.ijsolstr.2003.09.007

Ellen, 1962; Ohga et al., 1995), piecewise linear functions (e.g. Hwang, 1973), or as a non-linear function (e.g. Pines and Gerard, 1947; Malhorta et al., 1987; Navaneethakrishnan, 1988; Olhoff, 1974; Levy, 1996).

Chopra (1974) treated the free vibration of stepped plates as a composition of uniform domains, and the thickness was allowed to vary from domain to domain. The overall eigenvalue problem was formulated by assuming the boundary conditions and continuity conditions at the location of abrupt change of thickness. His work, however, contains an error in the continuity conditions for bending moment and shear force at the step, as indicated by Warburton (1975) and Yuan and Dickinson (1992). The Kantorovich extended method, including an exponential optimization parameter in the formulation, was utilized by Cortinez and Laura (1990) for analyzing the natural frequencies of stepped rectangular plates. The Rayleigh–Ritz method in association with a truncated double Fourier expansion, was applied by Bambill et al. (1991) to obtain the fundamental frequencies of simply supported stepped rectangular plates. However, it is believed that the assumption of continuity for the deflection function at the locations of abrupt change of thickness is a drawback of these earlier treatments. Harik et al. (1992) analyzed the vibration problem of rectangular stepped plates with correct continuity conditions. They modified the analytical strip method to allow step change in thickness in one direction. Yuan and Dickinson (1992) studied the vibration of simply supported stepped rectangular plates. They obtained the exact vibration frequencies for simply supported stepped rectangular plates by using the method proposed by Chopra (1974) with the correct continuity conditions for bending moment and shear force at the step. Recently, Cheung et al. (2000) have addressed the problem of excessive continuity by introducing a set of C^1 continuous longitudinal interpolation functions in the framework of the finite strip analysis to study buckling of rectangular stepped plates. The C^1 continuous functions are constructed by using the relevant beam vibration models with piecewise cubic polynomials. They have argued that these displacement functions possess both the advantages of fast convergence of harmonic functions as well as the appropriate order of continuity, and higher accuracy.

There are two important aspects that are worth paying attention in the on-going research on stepped plates. First, stepped plates in general do not admit analytical solutions. Most results reported are obtained by using numerical approaches with a few exceptions given by Chopra (1974), Yuan and Dickinson (1992), Eisenberger and Alexandrov (2000) for simply supported rectangular plates. It is highly important to have exact benchmark solutions so that numerical methods developed for analyzing non-uniform thickness plates can be validated on their convergence and accuracy. More recently, Xiang and Wang (2002) have introduced the Levy solution method to the problem of thin stepped plates having n-step variations in one direction parallel to the plate edges while the thickness is constant in the other direction. Another important aspect in treating the stepped plates concerns the theories used to model plates. As the thickness of a plate increases, it is crucial to include the effect of transverse shear deformation and rotary inertia in the analysis. Therefore, it is nature to consider the Mindlin first order shear deformable plate theory (Mindlin, 1951) or other higher-order plate theories for analyzing the buckling and free vibration of stepped plates. A treatment for stepped Mindlin plates was formulated by Ju et al. (1995) using a finite element approach. However, to our best knowledge, there are no exact solutions available in the literature for buckling and vibration of stepped Mindlin plates.

The objective of the present work is to fill this gap by providing exact solutions to the buckling and vibration of stepped rectangular Mindlin plates. By considering Mindlin plates with two opposite edges simply supported in the direction of the stepped variation, the general Levy type solution method in connection with a domain decomposition technique is employed to fulfill the objective.

This paper is organized as follows. Theoretical formulations are presented in Section 2. The Mindlin plate theory is employed and the general Levy type solution method is utilized to develop the analytical method for stepped Mindlin plates. Section 3 is devoted to results and discussions. Three types of problems, including buckling, free vibration of plates and vibration of plates subjected to inplane loads, are considered in the present study. Extensive first known exact buckling factors and frequency parameters are documented both for benchmarking new numerical algorithms and for engineering designs. This paper ends with a conclusion.

2. Theoretical formulation

Consider an isotropic, elastic, stepped rectangular plate of length aL, width L, modulus of elasticity E, Poisson's ratio v and shear modulus $G = E/[2(1+v)]$. The plate is of constant thicknesses in the y-direction and n-steps in the x-direction, with thickness h_i $(i = 1, 2, \ldots, n)$ for the ith step (see Fig. 1). The two edges of the plate parallel to the x-axis are assumed to be simply supported. The origin of the coordinate system is set at the centre of the bottom edge BC of the plate as shown in Fig. 1. The plate may be subjected to either a uni- or a bi-axial inplane compressive load. The problem at hand is to determine the critical buckling loads and the vibration frequencies for such an n-stepped rectangular plate.

The Mindlin first order shear deformation plate theory (Mindlin, 1951) is employed in this study. The plate is assumed to be simply supported on the two edges parallel to the x-axis. We take a typical step in the plate to derive the Levy type solution. The governing differential equations based on the Mindlin plate theory (Mindlin, 1951) for the ith step in harmonic vibration with inplane loads can be derived as:

$$\kappa^2 G h_i \left[\frac{\partial}{\partial x}\left(\frac{\partial w^i}{\partial x} + \theta_x^i \right) + \frac{\partial}{\partial y}\left(\frac{\partial w^i}{\partial y} + \theta_y^i \right) \right] - \beta N \frac{\partial^2 w^i}{\partial x^2} - \gamma N \frac{\partial^2 w^i}{\partial y^2} + \rho h_i \omega^2 w^i = 0 \tag{1}$$

$$D_i \left[\frac{\partial}{\partial x}\left(\frac{\partial \theta_x^i}{\partial x} + v \frac{\partial \theta_y^i}{\partial y} \right) \right] + \frac{(1-v)D_i}{2} \left[\frac{\partial}{\partial y}\left(\frac{\partial \theta_y^i}{\partial x} + \frac{\partial \theta_x^i}{\partial y} \right) \right] - \kappa^2 G h_i \left(\frac{\partial w^i}{\partial x} + \theta_x^i \right) + \frac{\rho h_i^3}{12} \omega^2 \theta_x^i = 0 \tag{2}$$

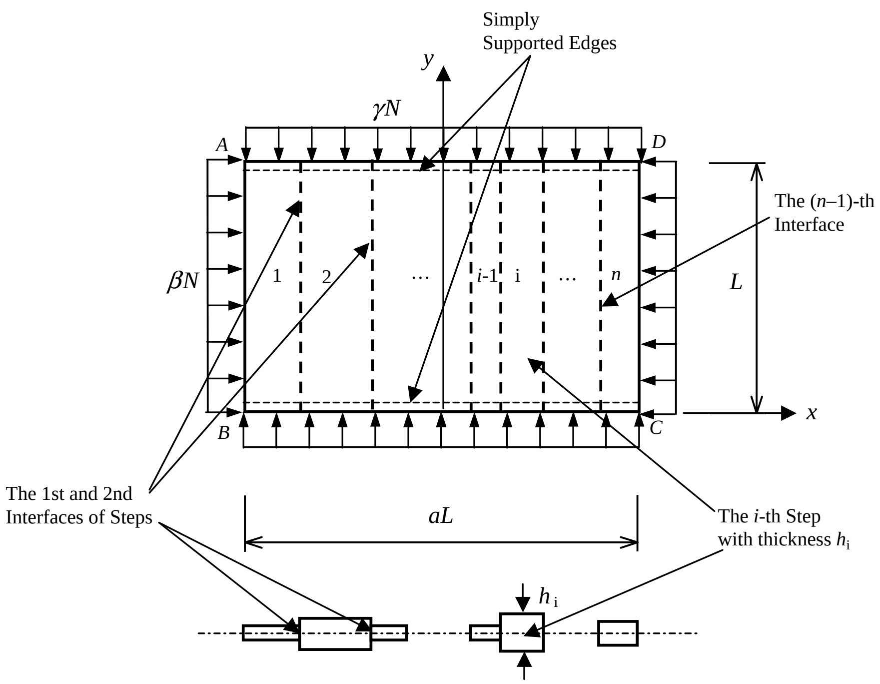


Fig. 1. Geometry and coordinate system for a multi-step rectangular Mindlin plate.

$$D_i\left[\frac{\partial}{\partial y}\left(\frac{\partial\theta_y^i}{\partial y}+v\frac{\partial\theta_x^i}{\partial x}\right)\right]+\frac{(1-v)D_i}{2}\left[\frac{\partial}{\partial x}\left(\frac{\partial\theta_y^i}{\partial x}+\frac{\partial\theta_x^i}{\partial y}\right)\right]-\kappa^2Gh_i\left(\frac{\partial w^i}{\partial y}+\theta_y^i\right)+\frac{\rho h_i^3}{12}\omega^2\theta_y^i=0 \tag{3}$$

where the sub- and super-script i $(=1,2,\ldots,n)$ denotes the ith step in the plate, κ^2 is the shear correction factor, $D_i=Eh_i^3/[12(1-v^2)]$ is the flexural rigidity of the ith step, ρ is the mass density of the plate, ω is the vibration frequency of the plate, w^i is the transverse displacement, θ_x^i and θ_y^i are rotations in the y and x directions, and β and γ are tracers that take the values of either 0 or 1 for different inplane load combinations.

The essential and natural boundary conditions for the two simply supported parallel edges (at $y=0$ and $y=L$) in the ith step are

$$w^i=0,\quad M_y^i=0,\quad \theta_x^i=0 \tag{4a–c}$$

where M_y^i is the bending moment and is defined by

$$M_y^i=D_i\left(\frac{\partial\theta_y^i}{\partial y}+v\frac{\partial\theta_x^i}{\partial x}\right) \tag{5}$$

The general Levy-type solution approach is employed to solve the governing differential equations for the ith step. The displacement fields can be expressed as

$$\begin{Bmatrix} w^i(x,y) \\ \theta_x^i(x,y) \\ \theta_y^i(x,y) \end{Bmatrix}=\begin{Bmatrix} \phi_w^i(x)\sin\frac{m\pi y}{L} \\ \phi_x^i(x)\sin\frac{m\pi y}{L} \\ \phi_y^i(x)\cos\frac{m\pi y}{L} \end{Bmatrix} \tag{6}$$

where $\phi_w^i(x)$, $\phi_x^i(x)$ and $\phi_y^i(x)$ are unknown functions to be determined. Eq. (6) satisfies the simply supported boundary conditions on edges at $y=0$ and $y=L$ as defined in Eqs. (4a)–(4c).

Substituting Eq. (6) into Eqs. (1)–(3), the following differential equation system can be derived:

$$(\boldsymbol{\psi}^i)'=\mathbf{H}^i\boldsymbol{\psi}^i \tag{7}$$

where $\boldsymbol{\psi}^i=[\phi_w^i\quad(\phi_w^i)'\quad\phi_x^i\quad(\phi_x^i)'\quad\phi_y^i\quad(\phi_y^i)']^{\mathrm{T}}$, the prime $'$ represents the derivative with respect to x and $\mathbf{H}^i$ is a 6×6 matrix with the following non-zero elements:

$$H_{12}^i=H_{34}^i=H_{56}^i=1 \tag{8}$$

$$H_{21}^i=\frac{-(m\pi/L)^2\kappa^2Gh_i+\gamma N(m\pi/L)^2+\rho h_i\omega^2}{-\kappa^2Gh_i+\beta N} \tag{9}$$

$$H_{24}^i=\frac{\kappa^2Gh_i}{-\kappa^2Gh_i+\beta N} \tag{10}$$

$$H_{25}^i=\frac{-(m\pi/L)\kappa^2Gh_i}{-\kappa^2Gh_i+\beta N} \tag{11}$$

$$H_{42}=\frac{\kappa^2Gh_i}{D_i}, \tag{12}$$

$$H_{46}=\frac{(m\pi/L)(1+v)}{2} \tag{13}$$

$$H_{43}^i=\frac{D_i(1-v)(m\pi/L)^2/2+\kappa^2Gh_i-\rho h_i^3\omega^2/12}{D_i} \tag{14}$$

$$H_{61}^{i} = \frac{(m\pi/L)\kappa^2 Gh_i}{[D_i(1-\nu)/2]} \tag{15}$$

$$H_{64}^{i} = -\frac{(m\pi/L)(1+\nu)}{1-\nu} \tag{16}$$

$$H_{65}^{i} = \frac{D_i(m\pi/L)^2 + \kappa^2 Gh_i - \rho h_i^3\omega^2/12}{[D_i(1-\nu)/2]} \tag{17}$$

A general solution of Eq. (7) can be obtained as

$$\boldsymbol{\psi}^{i} = \mathbf{e}^{\mathbf{H}^{i}x}\mathbf{c}^{i} \tag{18}$$

where $\mathbf{c}^i$ is a constant column vector that can be determined by the plate boundary conditions of the two edges parallel to the y-axis and/or the interface conditions between adjacent steps and $\mathbf{e}^{\mathbf{H}^i x}$ is the general matrix solution of Eq. (7). The detailed procedure in determining Eq. (18) has been given by Xiang et al. (1996).

Each of the two edges parallel to the y-axis may have the following edge conditions

$$M_x^i = 0, \quad M_{xy}^i = 0, \quad Q_x^i - \beta N\frac{\partial w^i}{\partial x} = 0, \quad \text{if the edge is free} \tag{19a–c}$$

$$w^i = 0, \quad M_x^i = 0, \quad \theta_y^i = 0, \quad \text{if the edge is simply supported} \tag{20a–c}$$

$$w^i = 0, \quad \theta_x^i = 0, \quad \theta_y^i = 0, \quad \text{if the edge is clamped} \tag{21a–c}$$

where i takes the value 1 or n, M_x^i, M_{xy}^i and Q_x^i are bending moment, twist moment and transverse shear force in the plate, respectively, and are defined by

$$M_x^i = D_i\left(\frac{\partial \theta_x^i}{\partial x} + \nu\frac{\partial \theta_y^i}{\partial y}\right) \tag{22}$$

$$M_{xy}^i = D_i\frac{1-\nu}{2}\left(\frac{\partial \theta_x^i}{\partial y} + \frac{\partial \theta_y^i}{\partial x}\right) \tag{23}$$

$$Q_x^i = \kappa^2 Gh_i\left(\frac{\partial w^i}{\partial x} + \theta_x^i\right) \tag{24}$$

Note that the free edge condition in Eq. (19c) involves the inplane load βN. The effect of this inplane force term on the buckling capacity of plates was discussed in an earlier paper by Xiang et al. (1996).

To ensure the continuity at the interface of adjacent steps, the essential and natural boundary conditions for the interface between the ith and the $(i+1)$th steps are defined as:

$$w^i = w^{i+1} \tag{25}$$

$$\theta_x^i = \theta_x^{i+1} \tag{26}$$

$$\theta_y^i = \theta_y^{i+1} \tag{27}$$

$$M_x^i = M_x^{i+1} \tag{28}$$

$$M_{xy}^{i} = M_{xy}^{i+1} \tag{29}$$

$$Q_{x}^{i} - \beta N \frac{\partial w^{i}}{\partial x} = Q_{x}^{i+1} - \beta N \frac{\partial w^{i+1}}{\partial x} \tag{30}$$

In view of Eq. (18), a homogeneous system of equations can be derived by implementing the boundary conditions of the plate along the two edges parallel to the y-axis (Eqs. (19)–(21)) and the interface conditions between two adjacent steps Eqs. (25)–(30) when assembling the steps to form the whole plate

$$\mathbf{K}\begin{Bmatrix} \mathbf{c}^{1} \\ \mathbf{c}^{2} \\ \vdots \\ \mathbf{c}^{(i-1)} \\ \mathbf{c}^{i} \\ \mathbf{c}^{(i+1)} \\ \vdots \\ \mathbf{c}^{n} \end{Bmatrix} = \{\mathbf{0}\} \tag{31}$$

The buckling load N (set $\omega = 0$) or the vibration frequency ω (set $N = 0$) may be determined when the determinant of $\mathbf{K}$ in Eq. (31) is equal to zero.

3. Results and discussions

The proposed analytical method is applied in this section to obtain exact solutions for buckling and vibration of stepped rectangular Mindlin plates. Three groups of results are presented in this section, namely, (1) buckling of stepped Mindlin plates; (2) free vibration of stepped Mindlin plates; and (3) vibration of stepped Mindlin plates subjected to inplane loads.

The critical buckling load N_{cr} and the vibration frequency ω are expressed in terms of a non-dimensional buckling factor $\lambda = N_{cr}L^2/(\pi^2 D_1)$ and a non-dimensional frequency parameter $\Lambda = (\omega L^2/\pi^2)\sqrt{\rho h_1/D_1}$, respectively, where h_1 and D_1 are the thickness and the flexural rigidity of the first step, respectively. For brevity, letters F, S and C are used to denote a free edge, a simply supported edge and a clamped edge, respectively. Since the Levy plates considered in the paper have the two edges parallel to the x-axis simply supported, we only need to use two letters to describe the plate boundary conditions on the two edges parallel to the y-axis. For instance, an *SF* plate has the edge *AB* simply supported and edge *DC* free (see Fig. 1). The Poisson's ratio $\upsilon = 0.3$ and the shear correction factor $\kappa^2 = 5/6$ are adopted for all cases in the paper.

3.1. Buckling of stepped Mindlin plates

For the buckling analysis of plates, we consider three inplane loading cases, namely, (1) uniaxial inplane compressive load in the x-direction ($\beta = 1$, $\gamma = 0$); (2) uniaxial inplane compressive load in the y-direction ($\beta = 0$, $\gamma = 1$); and (3) equi-biaxial inplane compressive loads ($\beta = 1$, $\gamma = 1$).

Table 1 presents the buckling factors λ generated by the present analytical approach and from Eisenberger and Alexandrov (2000) and Xiang and Wang (2002) for an *SS* rectangular plate with two even steps ($a = 2$, $b = 0.5$ in Fig. 2). The plate thickness ratio h_1/L is set to be 0.005 (quite thin) so that the results obtained from the present analytical method based on the Mindlin plate theory can be compared with the ones in Eisenberger and Alexandrov (2000) and Xiang and Wang (2002) based on the thin plate theory. Table 1 shows that the buckling solutions from the proposed analytical approach are in close agreement

Table 1
Comparison of buckling factors $\lambda = N_{cr}L^2/(\pi^2 D_1)$ for a two-step, *SS* rectangular plate subjected to uniaxial inplane load ($(\beta, \gamma) = (1, 0)$, $a = 2.0$, $b = 0.5$, $v = 0.25$, $h_1/L = 0.005$)

h_2/h_1	Sources		
	Present	Eisenberger and Alexandrov (2000)	Xiang and Wang (2002)
0.4	0.308264	0.8619	0.3083
0.6	1.02444	1.0245	1.0246
0.8	2.34385	2.3442	2.3442
1.0	3.99947	4.0000	4.0000
1.2	4.53154	4.5324	4.5325
1.4	4.66512	4.6663	4.6663
1.6	4.72798	4.7292	4.7292
1.8	4.76394	4.7652	4.7652
2.0	4.78646	4.7877	4.7878
2.2	4.80137	4.8026	4.8027

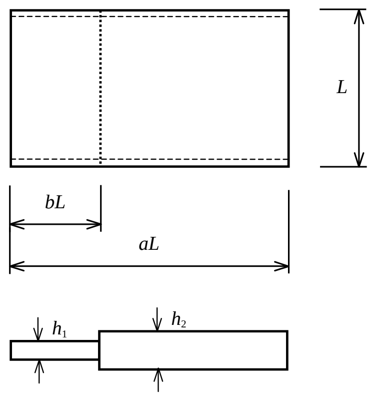


Fig. 2. A two-step rectangular Levy plate.

with the results from Eisenberger and Alexandrov (2000) and Xiang and Wang (2002) except for the case with $h_2/h_1 = 0.4$. The difference is attributed to the fact that Eisenberger and Alexandrov (2000) obtained the buckling load factor that corresponds to the third buckling mode while the authors obtained the correct value for the first buckling mode. Note that when using the present method to analyse Mindlin plates with small thickness ratios, high precision is required to perform the calculations, as indicated by Xiang et al. (1996). The comparison study confirms the correctness of the analytical method used in this paper.

Table 2 presents the buckling factors for the three symmetric Levy square plates (*SS*, *FF* and *CC* plates) with two-uneven steps (see Fig. 2). The step length parameter b varies from 0.3, 0.5 to 0.7. The step thickness ratios of the plates are set to be $h_2/h_1 = 1.2$ and 2.0. Two plate thickness ratios are considered, i.e. $h_1/L = 0.01$ (thin plates) and 0.1 (thick plates). As expected, we observe that the buckling factors decrease as the step length parameter b increases for all cases. The rate of decrease is more pronounced for plates subjected to the uniaxial inplane load in the y-direction ($\beta = 0$, $\gamma = 1$) and for thin plates ($h_1/L = 0.01$). The buckling factors increase as the step thickness ratio h_2/h_1 changes from 1.2 to 2.0. Again, the rate of increase is more significant for plates subjected to the uniaxial inplane load in the y-direction ($\beta = 0$, $\gamma = 1$) and for thin plates ($h_1/L = 0.01$). It is evident that the buckling factors decrease as the plate thickness ratio h_1/L increases, due to the influence of the transverse shear deformation in the plates.

Table 2
Buckling factors $\lambda = N_{cr}L^2/(\pi^2 D_1)$ for *SS*, *FF* and *CC* square Mindlin plates with two uneven steps

(β, γ)	h_2/h_1	B	*SS* (h_1/L)		*FF* (h_1/L)		*CC* (h_1/L)	
			0.01	0.1	0.01	0.1	0.01	0.1
(1,0)	1.2	0.3	5.73894	5.31202	2.54112	2.25565	10.1929	8.66030
		0.5	4.96157	4.62915	2.32618	2.06951	8.38620	7.25611
		0.7	4.50933	4.23342	2.25729	2.01377	7.59656	6.61825
	2.0	0.3	10.3908	8.65689	3.55553	3.04752	19.6097	14.3299
		0.5	7.73478	6.59061	2.53523	2.23441	13.7249	10.5611
		0.7	5.88004	5.22351	2.36079	2.09670	9.82583	8.03052
(0,1)	1.2	0.3	5.97715	5.56563	1.38233	1.32841	11.7662	9.29661
		0.5	5.19606	4.87277	1.24073	1.19488	9.91323	8.00557
		0.7	4.60111	4.33042	1.12765	1.08883	8.55673	7.01656
	2.0	0.3	16.4087	14.3564	4.19812	3.77306	33.8744	22.4755
		0.5	10.8319	9.53230	2.54567	2.31778	18.6525	13.1935
		0.7	7.54677	6.64562	1.76701	1.63940	11.4514	8.79030
(1,1)	1.2	0.3	2.95239	2.74867	1.25565	1.18527	5.73453	4.87078
		0.5	2.55839	2.39859	1.12401	1.06270	4.86501	4.21229
		0.7	2.28483	2.14983	1.04958	0.99841	4.39017	3.82521
	2.0	0.3	6.66970	5.77447	2.46874	2.14999	13.0149	9.87175
		0.5	4.70715	4.10376	1.54255	1.38579	8.72143	6.90426
		0.7	3.42855	3.04054	1.24278	1.13932	6.42577	5.25382

Table 3 shows the buckling factors for the three asymmetric Levy square plates (*SF*, *CF* and *CS* plates) with two-uneven steps. The buckling behaviour for these plates shows similar trends as the ones for the symmetric Levy square plates.

Fig. 3 presents the normalised buckling modal shapes in the x-direction for thick square plates ($h_1/L = 0.1$) with two-uneven steps. The step thickness ratio h_2/h_1 is set to be 2.0. As indicated in Fig. 3, the number of half waves in the y-direction is $m = 1$ for all cases except for *CC* plates subjected to uniaxial load in the y-direction ($\beta = 0$, $\gamma = 1$). The influence of step length parameter, load conditions and boundary conditions on the buckling modal shapes can be observed in Fig. 3.

The buckling of thick rectangular plates ($h_1/L = 0.1$) with two-, three- and four-even steps is considered in this study and the exact buckling factors for these plates are presented in Tables 4 and 5. The plate aspect ratios for the two-, three- and four-even-step plates are set to be $a = 2$, 3 and 4, respectively.

The step thickness variation for plates in Table 4 is moderate, i.e. $h_i/h_1 = 1 + (i-1) \times 0.1$, where i (=2, 3 and 4) refers to the ith step. We observe that the increase in the number of steps has insignificant effect on the buckling factors for *SS*, *FF*, *CC* and *CS* plates. It is due to the fact that the buckling behaviour of the plates is dominated by the first two steps of the plates. This is evident from the buckling modal shapes for the *SS* and *FF* plates shown in Fig. 4. For *SF* and *CF* plates, however, the buckling factors increase significantly as the number of steps increases, especially when plates are subjected to uniaxial inplane load in the x-direction ($\beta = 1$, $\gamma = 0$). The buckling modal shapes for the *SF* plates are shown in Fig. 4.

The step thickness variation for plates in Table 5 is large, where $h_i/h_1 = 1 + (i-1) \times 0.5$ and i (= 2, 3 and 4) refers to the ith step. It is observed that for all cases, the buckling factors have very small changes as the number of steps increases. The buckling behaviour of the plates is largely dependent on the behaviour of the first step of the plates. The interface between the first and the second steps acts as a clamped edge due to

Table 3
Buckling factors $\lambda = N_{cr}L^2/(\pi^2 D_1)$ for *SF*, *CF* and *CS* square Mindlin plates with two uneven steps

(β, γ)	h_2/h_1	b	*SF* (h_1/L)		*CF* (h_1/L)		*CS* (h_1/L)	
			0.01	0.1	0.01	0.1	0.01	0.1
(1,0)	1.2	0.3	3.96162	3.42241	4.06621	3.48887	7.73101	6.95682
		0.5	3.76688	3.27473	3.94129	3.38668	6.81856	6.17235
		0.7	3.45976	2.99624	3.62735	3.10394	5.81707	5.31784
	2.0	0.3	10.3339	8.50274	16.3249	12.2071	18.7209	14.0322
		0.5	7.71329	6.46078	11.8322	9.16984	12.2933	9.94679
		0.7	5.85874	5.05225	8.76927	6.79690	9.01789	7.53237
(0,1)	1.2	0.3	2.26032	2.13832	2.65021	2.46170	8.64779	7.65799
		0.5	2.08924	1.98366	2.48931	2.31813	7.65443	6.84520
		0.7	1.86540	1.77711	2.23296	2.08694	6.68322	6.03144
	2.0	0.3	8.52610	7.61282	10.2613	8.79378	25.1480	20.0522
		0.5	6.32485	5.76859	7.85241	6.95639	15.9189	13.0324
		0.7	4.39306	4.05635	5.50447	4.97696	10.9912	8.55983
(1,1)	1.2	0.3	1.73281	1.62035	1.89017	1.73822	4.11470	3.68878
		0.5	1.62461	1.52920	1.80704	1.66677	3.62700	3.27903
		0.7	1.46520	1.38533	1.64332	1.52051	3.13672	2.86277
	2.0	0.3	5.86099	5.22673	7.78923	6.63785	11.2214	8.93874
		0.5	4.18856	3.75571	5.78734	5.13002	7.08504	5.94800
		0.7	3.00021	2.72025	4.07133	3.65345	5.09024	4.30395

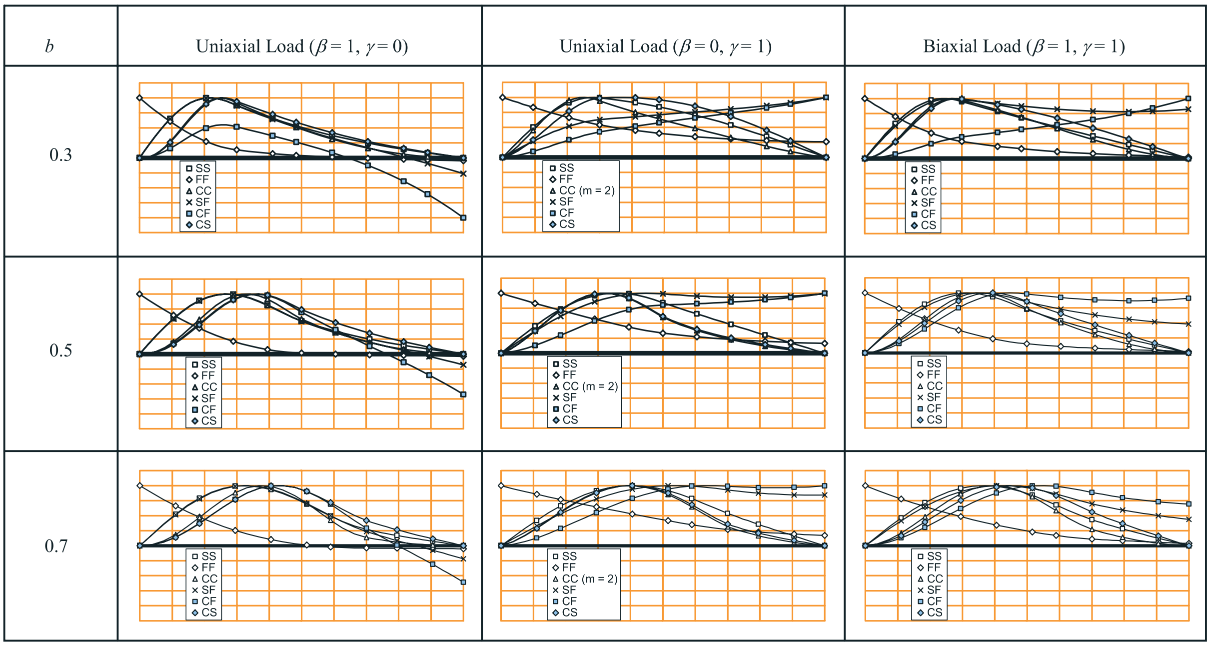
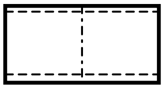
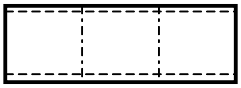
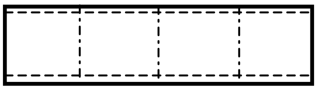


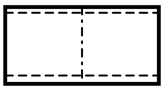
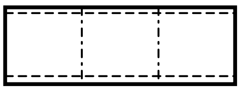
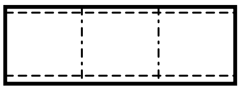
Fig. 3. Normalised buckling modal shapes in the x-direction for two-step thick square Mindlin plates ($h_1/L = 0.1$). The step thickness ratio $h_2/h_1 = 2.0$. The number of half waves in the y-direction $m = 1$ for all cases except for *CC* plates subjected to uniaxial load in the y-direction ($\beta = 0$, $\gamma = 1$).

Table 4
Buckling factors $\lambda = N_{cr}L^2/(\pi^2 D_1)$ for thick rectangular Mindlin plates ($h_1/L = 0.1$) having two-, three- and four-even steps

Cases	(β, γ)	*SS*	*FF*	*CC*	*SF*	*CF*	*CS*
	(1, 0)	4.11437	2.04985	4.96202	2.66856	2.66870	4.71285
	(0, 1)	1.72462	1.05327	2.08636	1.31254	1.35750	1.91397
	(1, 1)	1.36687	0.966345	1.60561	1.20687	1.23265	1.50748
Two-even-step plate ($h_2/h_1 = 1.1$)							
	(1, 0)	4.14349	2.05146	4.84721	3.40626	3.40634	4.83816
	(0, 1)	1.52322	1.08377	1.64247	1.44748	1.49862	1.61503
	(1, 1)	1.32560	0.968617	1.43472	1.31896	1.40517	1.42441
Three-even-step plate ($h_2/h_1 = 1.1$, $h_3/h_1 = 1.2$)							
	(1, 0)	4.14372	2.05146	4.84025	4.14361	4.25678	4.84021
	(0, 1)	1.49759	1.08560	1.57782	1.49188	1.56380	1.57440
	(1, 1)	1.32364	0.968653	1.41976	1.32351	1.41888	1.41928
Four-even-step plate ($h_2/h_1 = 1.1$, $h_3/h_1 = 1.2$, $h_4/h_1 = 1.3$)							

The plate aspect ratio is set to be $a = 2$, 3 and 4 for the two-, three- and four-step plates, respectively.

Table 5
Buckling factors $\lambda = N_{cr}L^2/(\pi^2 D_1)$ for thick rectangular Mindlin plates ($h_1/L = 0.1$) having two-, three- and four-even steps

Cases	(β, γ)	*SS*	*FF*	*CC*	*SF*	*CF*	*CS*
	(1, 0)	4.36305	2.06886	5.65752	4.36181	5.64081	5.64399
	(0, 1)	2.48421	1.24322	3.05614	2.31082	2.59967	2.93240
	(1, 1)	1.78076	1.01034	2.19611	1.76990	2.12950	2.15979
Two-even-step plate ($h_2/h_1 = 1.5$)							
	(1, 0)	4.36352	2.06905	5.64644	4.36351	5.64639	5.64642
	(0, 1)	2.44800	1.24962	2.86240	2.44573	2.85270	2.85816
	(1, 1)	1.78198	1.01072	2.16145	1.78192	2.16093	2.16109
Three-even-step plate ($h_2/h_1 = 1.5$, $h_3/h_1 = 2.0$)							
	(1, 0)	4.36352	2.06905	5.64643	4.36352	5.64643	5.64643
	(0, 1)	2.44751	1.24967	2.85696	2.44749	2.85685	2.85690
	(1, 1)	1.78199	1.01072	2.16112	1.78199	2.16112	2.16112
Four-even-step plate ($h_2/h_1 = 1.5$, $h_3/h_1 = 2.0$, $h_4/h_1 = 2.5$)							

The plate aspect ratio is set to be $a = 2$, 3 and 4 for the two-, three- and four-step plates, respectively.

the relatively large stiffness of the second step. It is noted that the boundary conditions at the right edge of the plates have limited effect on the buckling factors of the plates.

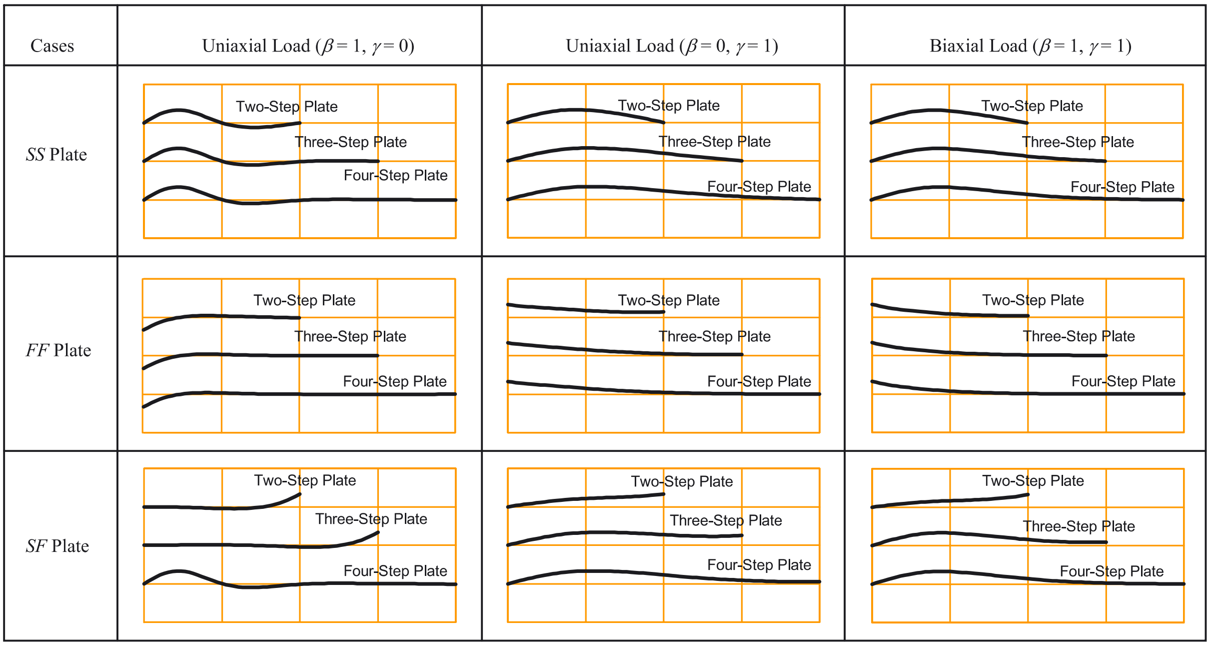


Fig. 4. Normalised buckling modal shapes in the *x*-direction for multi-step thick rectangular Mindlin plates ($h_1/L = 0.1$). The step thickness ratios are $h_2/h_1 = 1.1$, $h_3/h_1 = 1.2$, $h_4/h_1 = 1.3$. The number of half waves in the *y*-direction $m = 1$ for all cases.

3.2. Free vibration of stepped Mindlin plates

The free vibration of thick square and rectangular plates ($h_1/L = 0.1$) of multiple steps is studied. Table 6 presents frequency parameters obtained using the present analytical method and from Chopra (1974) and Yuan and Dickinson (1992) for a one-step *SS* square plate. The plate thickness ratio h_1/L is

Table 6
Comparison of frequency parameters $\Lambda = (\omega L^2/\pi^2)\sqrt{\rho h_1/D_1}$ for a one-step *SS* square plate ($a = 1$, $h_1/L = 0.005$)

b	h_2/h_1	Sources	Mode number					
			1	2	3	4	5	6
0.25	0.5	Present	1.29237	2.87075	2.89858	4.91918	5.41389	5.67875
		Yuan and Dickinson (1992)	1.29333	2.87183	2.89981	4.92249	4.41555	5.67965
	0.8	Present	1.70367	4.18625	4.19615	6.76425	8.24811	8.47926
		Yuan and Dickinson (1992)	1.70392	4.18715	4.19685	6.76611	8.25094	8.48212
0.75	0.5	Present	1.62893	4.04723	4.34043	6.86416	8.57067	8.70775
		Chopra (1974)	1.744	3.902	4.149	6.3875		
		Yuan and Dickinson (1992)	1.62903	4.04892	4.34142	6.86923	8.57562	8.71333
	0.8	Present	1.88915	4.68884	4.78228	7.55768	9.39662	9.62323
		Yuan and Dickinson (1992)	1.88936	4.68981	4.78334	7.56023	9.40069	9.62732

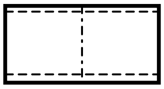
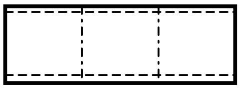
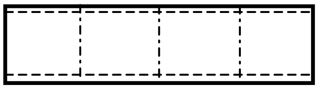
taken to be 0.005 (quite thin) so that the results obtained from the present analytical method based on the Mindlin plate theory can be compared with the ones by Chopra (1974) and Yuan and Dickinson (1992) based on the thin plate theory. Table 6 shows that the vibration solutions from the proposed analytical approach are in close agreement with the results from Yuan and Dickinson (1992), but are quite different from the ones by Chopra (1974). It is because the continuity conditions used in Chopra (1974) contain an error in the bending moment and shear force at the step (Warburton, 1975).

Table 7
Frequency parameters $\Lambda = (\omega L^2/\pi^2)\sqrt{\rho h_1/D_1}$ for two-step thick square plates ($h_1/L = 0.1$, $a = 1$)

Cases	b	h_2/h_1	Mode sequence					
			1	2	3	4	5	6
SS plate	0.3	1.2	2.16821	5.09925	5.16053	7.80544	9.45515	9.54106
		2.0	2.88021	6.61845	6.86149	10.0353	11.9260	11.9822
	0.5	1.2	2.09923	4.97480	4.99029	7.59152	9.16476	9.16799
		2.0	2.66931	6.08292	6.10788	8.96241	10.3367	10.5534
	0.7	1.2	2.03897	4.79434	4.81197	7.36864	8.84566	8.99825
		2.0	2.46138	5.18139	5.32766	8.27243	9.29564	9.78381
FF plate	0.3	1.2	1.08416	1.77060	3.87291	4.05850	4.82925	6.94544
		2.0	1.62332	2.59380	5.00928	5.04783	6.66252	8.74244
	0.5	1.2	1.04881	1.70530	3.69111	3.92461	4.74829	6.76250
		2.0	1.41550	2.22666	4.31018	4.38423	6.32847	8.23391
	0.7	1.2	1.01595	1.64382	3.55752	3.83455	4.64046	6.58684
		2.0	1.23780	1.99850	4.01290	4.03302	5.83686	7.20974
CC plate	0.3	1.2	2.97322	5.54320	6.53154	8.73154	9.74562	11.1061
		2.0	3.71756	7.25137	7.92843	10.8642	12.3024	13.4710
	0.5	1.2	2.91042	5.35570	6.41933	8.50642	9.36659	10.8003
		2.0	3.56140	6.57085	7.53462	9.80141	10.6934	11.9009
	0.7	1.2	2.87251	5.17106	6.23033	8.26739	9.02241	10.6139
		2.0	3.51296	5.75863	6.77794	9.12469	9.50895	11.2131
SF plate	0.3	1.2	1.33055	2.96662	4.47052	5.99403	6.22140	8.99964
		2.0	1.97237	3.86272	6.33455	7.86096	8.11507	11.4702
	0.5	1.2	1.29686	2.87350	4.37738	5.83582	6.09594	8.79312
		2.0	1.84078	3.64346	5.83563	7.09132	7.29871	10.2274
	0.7	1.2	1.25405	2.76959	4.23219	5.67076	5.82171	8.52389
		2.0	1.69084	3.20492	5.15055	6.17286	6.70715	9.26612
CF plate	0.3	1.2	1.42387	3.42249	4.51931	6.28844	6.97244	9.03284
		2.0	2.08749	4.35059	6.43445	8.18603	8.79152	11.9700
	0.5	1.2	1.39283	3.33184	4.43654	6.13135	6.86561	8.86318
		2.0	1.95388	4.10568	6.03368	7.66536	8.10880	10.6384
	0.7	1.2	1.34807	3.21561	4.29196	5.94871	6.56749	8.57633
		2.0	1.80127	3.70345	5.37044	6.86721	6.95441	9.45320
CS plate	0.3	1.2	2.52272	5.36143	5.81199	8.27266	9.67450	10.3003
		2.0	3.29206	7.14078	7.26431	10.5202	12.2794	12.8983
	0.5	1.2	2.46090	5.18374	5.70450	8.06984	9.31047	10.0393
		2.0	3.04378	6.47174	7.02677	9.54684	10.6845	11.3174
	0.7	1.2	2.39439	4.99173	5.52218	7.82108	8.96140	9.84228
		2.0	2.86712	5.64614	5.91064	8.70972	9.49299	10.7363

Table 7 presents the first six frequency parameters for the six Levy square plates with two-uneven steps. The step length parameter b varies from 0.3, 0.5 to 0.7 and the step thickness ratio h_2/h_1 is set to be 1.2 and 2.0. We observe that the frequency parameters decrease with increasing step length parameter b. It is due to

Table 8
Frequency parameters $\Lambda = (\omega L^2/\pi^2)\sqrt{\rho h_1/D_1}$ for thick rectangular Mindlin plates ($h_1/L = 0.1$) having two-, three- and four-even steps

Cases	Mode sequence	*SS*	*FF*	*CC*	*SF*	*CF*	*CS*
	1	1.27942	1.00775	1.40628	1.09754	1.11294	1.34137
	2	2.02032	1.21205	2.36807	1.52021	1.61641	2.18771
	3	3.20745	1.79614	3.74954	2.38807	2.58340	3.47450
	4	4.10821	2.76688	4.16078	3.66437	3.94840	4.14679
Two-even-step plate ($h_2/h_1 = 1.1$)	5	4.79993	3.79006	4.97444	4.00881	4.02386	4.89387
	6	4.81058	4.11368	5.48790	4.34346	4.39230	5.14049
	1	1.18445	1.02578	1.22566	1.13350	1.14443	1.21197
	2	1.53992	1.18737	1.66618	1.31553	1.35321	1.60516
	3	2.10553	1.45791	2.33324	1.73773	1.81922	2.22106
	4	2.87801	1.94178	3.19891	2.35923	2.48774	3.03977
Three-even-step plate ($h_2/h_1 = 1.1$,	5	3.84027	2.61324	4.10621	3.17627	3.34778	4.04191
$h_3/h_1 = 1.2$)	6	4.07642	3.46989	4.23744	4.07351	4.10116	4.10552
	1	1.16651	1.02788	1.18944	1.15831	1.17399	1.18676
	2	1.39625	1.21898	1.45310	1.28739	1.30424	1.43091
	3	1.72969	1.36078	1.84237	1.51968	1.56309	1.79020
	4	2.18838	1.64878	2.35951	1.89550	1.96562	2.27869
Four-even-step plate ($h_2/h_1 = 1.1$,	5	2.76928	2.05854	2.99480	2.38579	2.48332	2.88594
$h_3/h_1 = 1.2$, $h_4/h_1 = 1.3$)	6	3.46124	2.57860	3.73492	2.99339	3.11628	3.60263

The plate aspect ratio is set to be $a = 2$, 3 and 4 for the two-, three- and four-step plates, respectively.

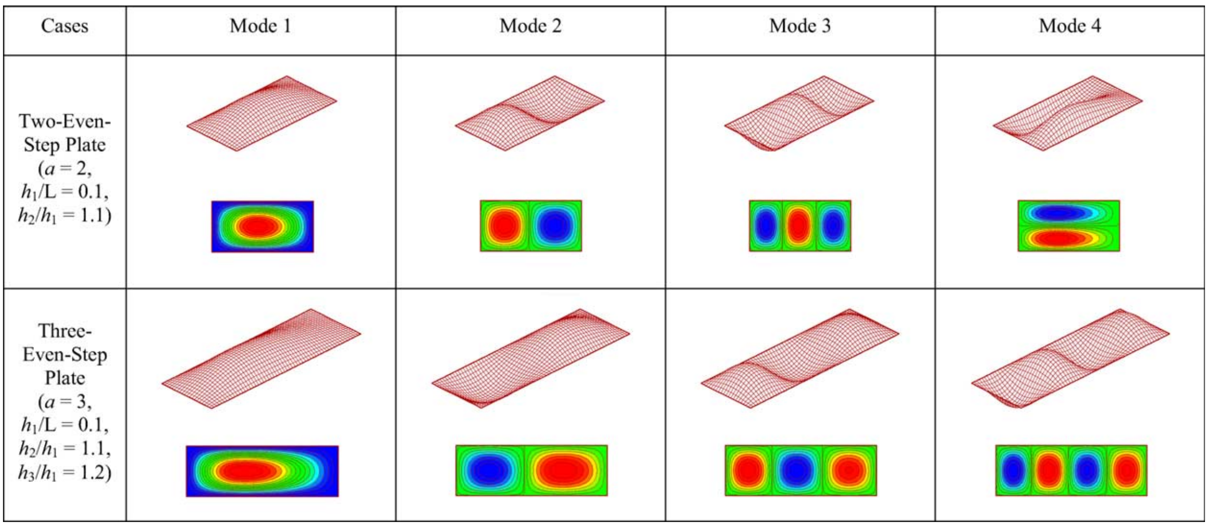


Fig. 5. Modal shapes and contours for the first four modes of two- and three-even-step *SS* rectangular plates.

the decrease of the overall stiffness of the plate as the step length parameter b increases. On the other hand, the frequency parameters increase as the step thickness ratio h_2/h_1 changes from 1.2 to 2.0.

Table 8 presents the first six frequency parameters for thick rectangular plates with two-, three- and four-even steps. The step thickness variation for plates is $h_i/h_1 = 1 + (i-1) \times 0.1$, where i (= 2, 3 and 4) refers to the ith step. The plate aspect ratios for the two-, three- and four-even-step plates are taken as $a = 2$, 3 and 4, respectively. Unlike the buckling counterpart, where the increase of the number of steps has limited effect on the buckling factors, the frequency parameters change significantly as the number of steps increases. The typical vibration modal shapes and contours for the first four modes of the *SS* and *SF* rectangular plates are presented in Figs. 5 and 6. We can clearly observe the effect of steps on the modal shapes of these plates.

3.3. Vibration of stepped Mindlin plates subjected to inplane loads

The proposed analytical method can be used to study the vibration of Mindlin plates subjected to inplane loads. We have chosen the *SS* and *FF* thick rectangular plates ($h_1/L = 0.1$) of two-even steps

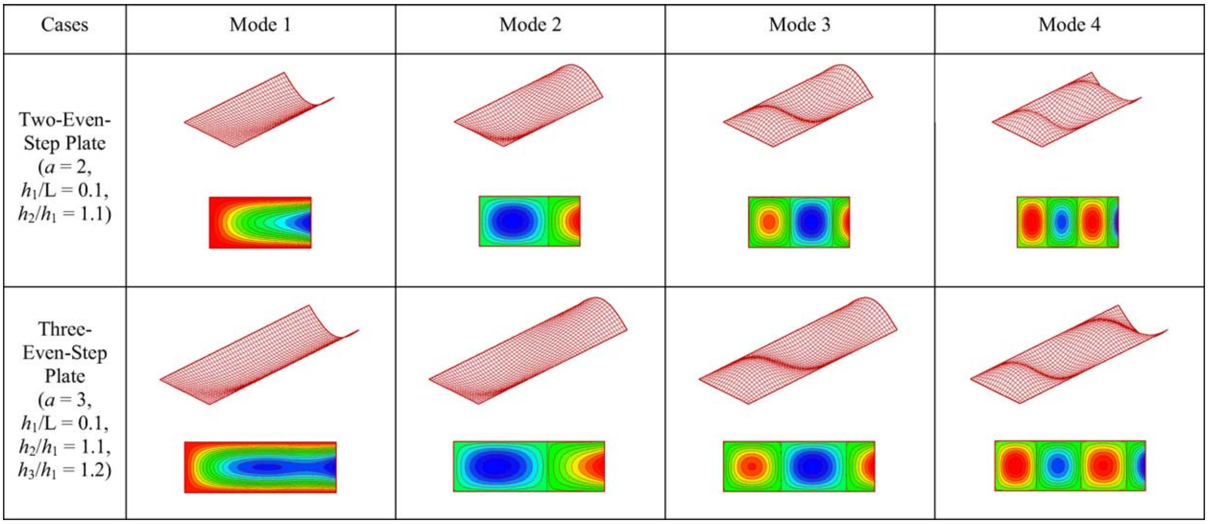


Fig. 6. Modal shapes and contours for the first four modes of two- and three-even-step *SF* rectangular plates.

Table 9
Frequency parameters $\Lambda = (\omega L^2/\pi^2)\sqrt{\rho h_1/D_1}$ for *SS* and *FF* thick rectangular Mindlin plates ($h_1/L = 0.1$) having two-even steps

Case	Mode sequence	*SS* (N/N_{cr})				*FF* (N/N_{cr})			
		−0.9	−0.5	0.5	0.9	−0.9	−0.5	0.5	0.9
	1	1.58694	1.45882	1.06152	0.623406	1.01845	1.01551	0.968869	0.595461
	2	2.75113	2.45354	1.46477	0.933386	1.39835	1.32104	1.08179	0.973939
	3	4.22193	3.82348	2.43790	1.57291	2.27423	2.08030	1.42549	1.08352
	4	4.25172	4.17267	3.92716	3.05541	3.46783	3.18084	2.25355	1.73172
Two-even-step plate	5	5.15168	5.00277	4.03867	3.97644	3.81302	3.80417	3.55279	3.01205
($h_2/h_1 = 1.1$)	6	6.06134	5.53649	4.61130	4.44673	4.16736	4.14393	3.77044	3.74752

The plate aspect ratio is set to be $a = 2$ and the plate is subjected to uniaxial inplane load in the x-direction ($\beta = 1$, $\gamma = 0$).

($h_2/h_1 = 1.1$) to demonstrate the application of the method. The plates are subjected to uniaxial inplane load in the x-direction ($\beta = 1$, $\gamma = 0$). The inplane load ratio N/N_{cr}, where N_{cr} is the critical buckling load and may be obtained from Table 4, varies from −0.9, −0.5, 0.5 to 0.9. The negative value of N/N_{cr} denotes that the inplane load N is a tensile force. The frequency parameters for the two plates are presented in Table 9. We observe that the frequency parameters for the plates decrease as the inplane load ratio N/N_{cr} varies from −0.9 to 0.9.

4. Conclusions

This paper presents an analytical approach for studying the buckling and vibration behaviour of rectangular Mindlin plates with multiple steps. The Levy solution method is employed in connection with the domain decomposition technique that is used to cater for the step variation in the plates. Presented in the paper are the first-known exact solutions for buckling and vibration of stepped rectangular Mindlin plates with two opposite edges simply supported and the remaining two edges being either free, simply supported or clamped. The influence of the step length ratios, step thickness ratios and the number of steps on the buckling and vibration behaviour of square and rectangular Mindlin plates is investigated. The authors believe that the presented exact solutions for buckling and vibration of the stepped Mindlin plates are very valuable as they may serve as benchmark results for future researches in this area.

Acknowledgements

This work was supported by the University of Western Sydney and by the National University of Singapore.

References

Bambill, D.V., Laura, P.A.A., Bergmann, A., Carnicer, R., 1991. Fundamental frequency of transverse vibration of symmetrically stepped simply supported rectangular plates. Journal of Sound and Vibration 150 (1), 167–169.

Cheung, Y.K., Au, F.T.K., Zheng, D.Y., 2000. Finite strip method for the free vibration and buckling analysis of plates with abrupt changes in thickness and complex support conditions. Thin-Walled Structures 36, 89–110.

Chopra, I., 1974. Vibration of stepped thickness plates. International Journal of Mechanical Sciences 16, 337–344.

Cortinez, V.H., Laura, P.A.A., 1990. Analysis of vibrating rectangular plates of discontinuously varying thickness by means of the Kantorovich extended method. Journal of Sound and Vibration 137 (3), 457–461.

Eisenberger, M., Alexandrov, A., 2000. Stability analysis of stepped thickness plates. In: Papadrakakis, M., Samartin, A., Onate, E. (Eds.), Computational Methods for Shell and Spatial Structures, IASS-IACM 2000, ISASR-NTUA, Athens, Greece.

Guo, S.J., Keane, A.J., Moshrefi-Torbati, M., 1997. Vibration analysis of stepped thickness plates. Journal of Sound and Vibration 204, 645–657.

Harik, I.E., Liu, X., Balakrishnan, N., 1992. Analytical solution to free vibration of rectangular plates. Journal of Sound and Vibration 153 (1), 51–62.

Hwang, S.S., 1973. Stability of plates with piecewise varying thickness. Journal of Applied Mechanics 40, 1127–1129.

Ju, F., Lee, H.P., Lee, K.H., 1995. Free vibration of plates with stepped variations in thickness on non-homogeneous elastic foundations. Journal of Sound and Vibration 183, 533–545.

Lam, K.Y., Amrutharaj, G., 1995. Natural frequencies of rectangular stepped plates using polynomial functions with subsectioning. Applied Acoustics 44, 325–340.

Levy, R., 1996. Rayleigh-Ritz optimal design of orthotropic plates for buckling. Structural Engineering and Mechanics 4, 541–552.

Malhorta, S.K., Ganesan, N., Veluswami, M.A., 1987. Vibrations of orthotropic square plates having variable thickness (parabolic variation). Journal of Sound and Vibration 119, 184–188.

Mindlin, R.D., 1951. Influence of rotatory inertia and shear on flexural motions of isotropic, elastic plates. Journal of Applied Mechanics, ASME 18, 31–38.

Navaneethakrishnan, P.V., 1988. Buckling of nonuniform plates: Spline method. Journal of Engineering Mechanics, ASCE 114, 893–898.

Ohga, M., Shigematsu, T., Kawaguchi, K., 1995. Buckling analysis of thin-walled members with variable thickness. Journal of Structural Engineering ASCE 121, 919–924.

Olhoff, N., 1974. Optimal design of vibrating rectangular plates. International Journal of Solids and Structures 10, 93–109.

Pines, S., Gerard, G., 1947. Instability analysis and design of an efficiently tapered plate under compressive loading. Journal of the Aeronautical Science 14, 500–594.

Warburton, G.B., 1975. Comment on "Vibration of stepped plates" by I. Chopra. International Journal of Mechanical Sciences 117, 239.

Wittrick, W.H., Ellen, C.H., 1962. Buckling of tapered rectangular plates in compression. Aeronautical Quarterly 13, 308–326.

Xiang, Y., Liew, K.M., Kitipornchai, S., 1996. Exact buckling solutions for composite laminates: proper free edge conditions under in-plane loadings. Acta Mechanica 117, 115–128.

Xiang, Y., Wang, C.M., 2002. Exact buckling and vibration solutions for stepped rectangular plates. Journal of Sound and Vibration 250, 503–517.

Yuan, J., Dickinson, S.M., 1992. The flexural vibration of rectangular plate systems approached by using artificial springs in the Rayleigh–Ritz method. Journal of Sound and Vibration 159, 39–55.